

## CONTRIBUTIONS TO THE THEORY OF INTERPOLATION OF OPERATIONS II

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### 1. Introduction

In this paper we shall discuss the interpolation of operations on intermediate spaces. Our method is the so-called real method. Our purpose is to treat the critical case which appears in singular integral operators. If we consider for example the Hilbert transform  $\tilde{f}$  of function  $f$  of the class  $L \log^+ L(\pm\infty, \infty)$ ,  $\tilde{f}$  exist a.e. but the only local integrability holds. Then we shall discuss their integral estimation on the whole space.

The intermediate space between two Banach spaces was introduced by W.A.J. Luxemburg [6, 7]. This is defined as follows. Given a topological vector space  $V$  and two Banach spaces  $A_1$  and  $A_2$  which are contained and continuously embedded in  $V$ . If  $f$  is an element of  $A_i$  ( $i=1, 2$ ), we denote its norm by  $\|f\|_{[A_i]}$  ( $i=1, 2$ ). We shall consider the space  $A_1+A_2$  and introduce in it the norm

$$\|f\|_{[A_1+A_2]} = \inf (\|g\|_{[A_1]} + \|h\|_{[A_2]})$$

where the infimum is taken over all pairs  $g \in A_1$  and  $h \in A_2$  such that  $f=g+h$ , then  $A_1+A_2$  also becomes a Banach space. Since  $A_1$  and  $A_2$  are continuously embedded in  $V$ , it is evident that  $A_1+A_2$  is also continuously embedded in  $V$ .

In what follows we shall consider totally  $\sigma$ -finite measure space  $(R, \mu)$  and the space  $V$  of equivalent classes of real valued measurable functions on  $R$ . The equivalent relation here is that of coincidence almost everywhere. If in  $V$  we introduce a topology of convergence in measure on sets of finite measure,  $V$  becomes a topological vector space. If we take as the interpolation pair  $A_1=L^1_\mu$ ,  $A_2=L^p_\mu$  ( $1 < p < \infty$ ) then these are continuously embedded in  $V$ . We shall also consider another measure space  $(S, \nu)$ .

Let us consider operation  $T$  which transforms measurable functions on  $R$  to those on  $S$ . The operation  $T$  is called quasi-linear if

- (i)  $T(f_1+f_2)$  is uniquely defined whenever  $Tf_1$  and  $Tf_2$  are defined and

$$|T(f_1+f_2)| \leq \kappa(|Tf_1| + |Tf_2|)$$

where  $\kappa$  is a constant independent of  $f_1$  and  $f_2$ ,

(ii)  $T(cf)$  is uniquely defined whenever  $Tf$  is defined and

$$|T(cf)| = |c| |Tf|$$

for all scalars  $c$ .

We say that the operation  $T$  is of type  $(a, b)$ ,  $1 \leq a \leq b \leq \infty$ , if  $Tf$  is defined for each  $f \in L_\mu^a(R)$  and belongs to  $L_\mu^b(S)$  such that

$$(1.1) \quad ||Tf|| [L_\nu^b] \leq M ||f|| [L_\mu^a]$$

where  $M$  is a constant independent of  $f$ . The least admissible value of  $M$  in (1.1) is called the  $(a, b)$ -norm of operation  $T$ . Next we shall define the weak type  $(a, b)$  of operations. Suppose first that  $1 \leq b < \infty$ . Given any  $r > 0$ , denote by  $E_r = E_r[Tf]$  the set of points of the space  $S$  where  $|Tf| > r$ , and write  $\nu(E_r)$  for the  $\nu$ -measure of the set  $E_r$ . We say that the operation  $T$  is of weak type  $(a, b)$  if

$$(1.2) \quad \nu(E_r[Tf]) \leq \left( \frac{M}{r} ||f|| [L_\mu^a] \right)^b$$

where  $M$  is a constant independent of  $f$ . The least admissible value of  $M$  in (1.2) is called the weak  $(a, b)$ -norm of operation  $T$ . It is clear that being of type  $(a, b)$  implies being of weak type  $(a, b)$ . We shall define weak type  $(a, \infty)$  as identical with type  $(a, \infty)$ . Hence  $T$  is of weak type  $(a, \infty)$  if

$$(1.3) \quad \text{ess. sup} ||Tf|| \leq M ||f|| [L_\mu^a]$$

Beside the space  $L_\mu^a$  we shall need the space  $L_\mu^a(\log^+ L_\mu)^c$  which consist of functions to be  $\mu$ -measurable and such that

$$||f|| [L_\mu^a(\log^+ L_\mu)^c] = \left( \int_R |f|^a (1 + (\log^+ |f|)^c) d\mu \right)^{1/a} < \infty.$$

This is not a Banach space but we shall use conveniently the same notations  $L_{\mu^1}^a(\log^+ L_\mu)^{a_1/b_1} + L_{\mu^2}^a$  as in the preceding case, which consist of functions to be  $\mu$ -measurable and such that

$$||f|| [L_{\mu^1}^a(\log^+ L_\mu)^{a_1/b_1} + L_{\mu^2}^a] = \inf (||h|| [L_{\mu^1}^a(\log^+ L_\mu)^{a_1/b_1}] + ||g|| [L_{\mu^2}^a]) < \infty$$

where the infimum is taken over all pairs  $h \in L_{\mu^1}^a(\log^+ L_\mu)^{a_1/b_1}$  and  $g \in L_{\mu^2}^a$  such that  $f = g + h$

In our preceding paper [4] we proved that

**Theorem A.** Suppose that a quasi-linear operation  $T$  is of weak type  $(1, 1)$  and of type  $(p, p)$  for some  $1 < p < \infty$ . Then we have

$$\begin{aligned} & \int_{|Tf| \leq 1} |Tf|^p d\nu + \int_{|Tf| > 1} |Tf| d\nu \\ & \leq K \left( \int_{|f| \leq 1} |f|^p d\mu + \int_{|f| > 1} |f| (1 + \log^+ |f|) d\mu \right) \end{aligned}$$

where  $K$  is a constant independent of  $f$ .

We can grow it up to the following form

**Theorem 1.** Under the hypotheses of theorem A, we have  $Tf \in L_\nu + L_\mu^p$  for any  $f \in L_\mu \log^+ L_\mu + L_\mu^p$  and

$$||Tf|| \leq KM^p ||f|| \{1 + \log^+ ||f||\}^{-1}$$

where  $K$  is a constant depending on  $p, \kappa$  but not on  $M_1, M_2$  and  $f$ ;  $M = \max(M_1, M_2, 1)$

Furthermore we shall prove the following theorem which gives an answer to the conjecture of [4: p. 148~9].

**Theorem 2.** Let us write  $\alpha_i = 1/a_i, \beta_i = 1/b_i$  ( $i=1, 2$ ). Let  $(\alpha_1, \beta_1)$  and  $(\alpha_2, \beta_2)$  be any two point of the triangle

$$\Delta; 0 < \beta \leq \alpha \leq 1$$

such that  $\beta_1 \neq \beta_2$ . Let us suppose that a quasi-linear operation  $\tilde{f} = Tf$  is of weak type  $(1/\alpha_1, 1/\beta_1)$  and of type  $(1/\alpha_2, 1/\beta_2)$  with norms  $M_1$  and  $M_2$  respectively. Then if  $\alpha_1 > \alpha_2$  and  $\beta_1 > \beta_2$  we have  $Tf \in L_\nu^{b_1} + L_\nu^{b_2}$  for any  $f \in L_\mu^{a_1} (\log^+ L_\mu)^{a_1/b_1} + L_\mu^{a_2}$  and

$$||Tf|| \leq KM^{b_2/b_1} ||f|| (1 + (\log^+ ||f||)^{a_1/b_1})^{1/a_1}$$

where  $K$  is a constant depending on  $\alpha_1, \alpha_2, \beta_1, \beta_2, \kappa$  and not on  $M_1, M_2$  and  $f$ ;  $M = \max(M_1, M_2, 1)$ .

**Corollary 1.** Let us write

$$\begin{cases} a_\theta = (1-\theta)a_1 + \theta a_2 \\ b_\theta = (1-\theta)b_1 + \theta b_2 \end{cases} \quad (0 < \theta \leq 1).$$

Then under the hypotheses of theorem 2, we have  $Tf \in L_\nu^{b_1} + L_\nu^{b_2}$  for any  $f \in L_\mu^{a_1} (\log^+ L_\mu)^{a_1/b_1} + L_\mu^{a_2}$  and

$$||Tf|| \leq K_\theta M^{b_2/b_1} ||f|| \{1 + (\log^+ ||f||)^{a_1/b_1}\}^{1/a_1}.$$

where  $K_\theta$  is a constant depending on  $\theta, \alpha_1, \alpha_2, \beta_1, \beta_2$ , but not on  $M_1, M_2, f$ .

Since by the Marcinkiewicz-Zygmund theorem it follows that the operation

$T$  is of type  $(a_\theta, b_\theta)$  by the hypotheses of theorem 2 and so we can derive corollary 1 immediately.

Next T. Kawata [3] proved the following theorem

**Theorem B.** *Let  $f(x)$  be a measurable function of the class  $L(-\infty, \infty)$ . Then its Hillbert transform*

$$\tilde{f}(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{f(y)}{x-y} dy$$

*exists and satisfies the following inequality*

$$\int_{-\infty}^{\infty} \frac{|\tilde{f}(x)|}{1 + |\log |\tilde{f}(x)||^{1+\varepsilon}} dx \leq M \int_{-\infty}^{\infty} |f(x)| dx$$

*where  $\varepsilon$  is any positive number and  $M$  is a constant independent of  $f$ .*

Furthermore the  $\varepsilon > 0$  can not be omitted as a counter example shows.

Here we introduce the space  $L_{\nu^1}^{b_1}/(\log^+ L_{\nu})^{1+\varepsilon}$  which consist of functions such that

$$||f|| [L_{\nu^1}/(\log^+ L_{\nu})^{1+\varepsilon}] = \left( \int_s \frac{|f(x)|^{b_1}}{1 + (\log^+ |f(x)|)^{1+\varepsilon}} d\nu \right)^{1/b_1} < \infty$$

and also the space  $L_{\nu^1}^{b_1}/(\log^+ L_{\nu})^{1+\varepsilon} + L_{\nu^2}^{b_2}$  which consist of functions

$$||f|| [L_{\nu^1}/(\log^+ L_{\nu})^{1+\varepsilon} + L_{\nu^2}^{b_2}] = \inf \{ ||h|| [L_{\nu^1}/(\log^+ L_{\nu})^{1+\varepsilon}] + ||g|| [L_{\nu^2}^{b_2}] \} < \infty$$

where the infimum is taken over all pairs  $h \in L_{\nu^1}/(\log^+ L_{\nu})^{1+\varepsilon}$  and  $g \in L_{\nu^2}^{b_2}$  such that  $f = g + h$ .

Then if we discuss his result from the stand point of views of the theory of interpolation of operation, we shall prove

**Theorem 3.** *Under the hypotheses of theorem 2, we have  $Tf \in L_{\nu^1}^{b_1}/(\log^+ L_{\nu})^{1+\varepsilon} + L_{\nu^2}^{b_2}$  for any  $f \in L_{\mu^1}^{a_1} + L_{\mu^2}^{a_2}$  and*

$$||Tf|| \leq KM^{b_2/b_1} ||f|| \{1 + (\log^+ ||f||^{-1})^{1+\varepsilon/b_1}\}$$

*where  $\varepsilon$  is any positive real number,  $K$  is a constant independent of  $f$  and  $M = \max(M_1, M_2, 1)$ .*

**Corollary 2.** *Under the hypotheses of theorem 3, we have  $Tf \in L_{\nu^1}^{b_1}/(\log^+ L_{\nu})^{1+\varepsilon} + L_{\nu^2}^{b_2}$  for any  $f \in L_{\mu^1}^{a_1} + L_{\mu^2}^{a_2}$  and*

$$||Tf|| \leq K_\theta M^{b_2/b_1} ||f|| \{1 + (\log^+ ||f||^{-1})^{1+\varepsilon/b_1}\}.$$

Furthermore we shall study the case  $\theta = 0$ . Let us denote by  $L_{\nu^1}^{b_1}/(\log^+ L_{\nu}^{-1})^{1+\varepsilon}$  the set of functions such that  $f$  is  $\nu$ -measurable and

$$||f|| [L_{\nu}^{b_1}/(\log^+ L_{\nu}^{-1})^{1+\varepsilon}] = \left( \int_s \frac{|f|^{b_1}}{1 + (\log^+ |f|^{-1})^{1+\varepsilon}} d\nu \right)^{1/b_1} < \infty.$$

Then we have

**Theorem 4.** *Let us suppose that the operation  $T$  is of weak type  $(a_1, b_1)$ ,  $1 \leq a_1, b_1 < \infty$  then we have  $Tf \in L_{\nu}^{b_1}/(\log^+ L_{\nu})^{1+\varepsilon} + L_{\nu}^{b_1}/(\log^+ L_{\nu}^{-1})^{1+\varepsilon}$  for any  $f \in L_{\mu}^{a_1}$  and*

$$||Tf|| [L_{\nu}^{b_1}/(\log^+ L_{\nu})^{1+\varepsilon} + L_{\nu}^{b_1}/(\log^+ L_{\nu}^{-1})^{1+\varepsilon}] \leq KM_1 ||f|| [L_{\mu}^{a_1}]$$

where  $K$  is a constant independent of  $f$  and  $\varepsilon$  is any positive number.

The case  $a_1 = b_1 = 1$  and  $\theta = 0$  corresponds to the theorem of T. Kawata. Finally we shall prove another type of theorems which correspond to the theorem of N. Kolmogoroff [5] about conjugatae functions and the theorem of A.P. Calderon-A. Zygmund [1] about singular integral operators.

**Theorem 5.** *Under the hypothesis of theorem 2, we have  $Tf \in L_{\nu}^{b_1-\varepsilon} + L_{\nu}^{b_2}$  for any  $f \in L_{\mu}^{a_1} + L_{\mu}^{a_2}$  and*

$$||Tf|| [L_{\nu}^{b_1-\varepsilon} + L_{\nu}^{b_2}] \leq KM^{b_2/b_1-\varepsilon} ||f|| [L_{\mu}^{a_1} + L_{\mu}^{a_2}]$$

where  $\varepsilon$  is any positive number,  $K$  is a constant independent of  $f$  and  $M = \max(M_1, M_2, 1)$ .

**Corollary 3.** *Under the hypothesis of theorem 5, we have  $Tf \in L_{\nu}^{b_1-\varepsilon} + L_{\nu}^{b_2\theta}$  for any  $f \in L_{\mu}^{a_1} + L_{\mu}^{a_2\theta}$  and*

$$||Tf|| [L_{\nu}^{b_1-\varepsilon} + L_{\nu}^{b_2\theta}] \leq K_{\theta} M^{b_2/b_1-\varepsilon} ||f|| [L_{\mu}^{a_1} + L_{\mu}^{a_2\theta}] \quad (0 < \theta \leq 1)$$

Corresponding to the case  $\theta = 0$ , we have

**Theorem 6.** *Under the hypothesis of theorem 4, we have  $Tf \in L_{\nu}^{b_1-\varepsilon} + L_{\nu}^{b_1+\varepsilon}$  for any  $f \in L_{\mu}^{a_1}$  and*

$$||Tf|| [L_{\nu}^{b_1-\varepsilon} + L_{\nu}^{b_1+\varepsilon}] \leq KM_1^{b_1/b_1-\varepsilon} ||f|| [L_{\mu}^{a_1}]$$

where  $K$  is a constant independent of  $f$ .

Furthermore we shall prove

**Theorem 7.** *Under the hypotheses of theorem 2 except that  $\alpha_1 > \alpha_2$  and  $\beta_1 < \beta_2$ , we have  $Tf \in L_{\nu}^{b_1+\varepsilon} + L_{\nu}^{b_2}$  for any  $f \in L_{\mu}^{a_1} + L_{\mu}^{a_2}$  and*

$$||Tf|| [L_{\nu}^{b_1+\varepsilon} + L_{\nu}^{b_2}] \leq KM ||f|| [L_{\mu}^{a_1} + L_{\mu}^{a_2}]$$

where  $\varepsilon$  is any positive number,  $K$  is a constant independent of  $f$  and  $M = \max(M_1, M_2, 1)$ .

**Corollary 4.** *Let us write*

$$\begin{cases} a_\theta = (1-\theta)a_1 + \theta a_2 \\ b_\theta = (1-\theta)b_1 + \theta b_2 \end{cases} \quad (0 < \theta \leq 1)$$

*Then under the hypotheses of theorem 7, we have  $Tf \in L_{\nu}^{b_1+\varepsilon} + L_{\nu}^{b_\theta}$  for any  $f \in L_{\mu}^{a_1} + L_{\mu}^{a_\theta}$  and*

$$||Tf||[L_{\nu}^{b_1+\varepsilon} + L_{\nu}^{b_\theta}] \leq K_\theta M ||f||[L_{\mu}^{a_1} + L_{\mu}^{a_\theta}].$$

As an application we shall consider some singular integral operators. One of them is that of the Hilbert-Calderon-Zygmund which is defined as follows.

$$\tilde{f}(x) = P.V. \int_{\mathbb{R}^n} K(x-y) f(y) dy$$

where the kernel  $K(x)$  has the form

$$K(x) = |x|^{-n} \Omega(x'), \quad x' = x/|x|.$$

Let us denote by  $\Sigma$  the unit sphere on which the  $\Omega(x')$  is defined. Let us also denote by  $\omega(\delta)$  the modulus of continuity of  $\Omega(x')$ ,

$$|\Omega(x') - \Omega(y')| \leq \omega(x' - y').$$

Let us suppose that

$$(i) \quad \int_{\Sigma} \Omega(x') dx = 0$$

(ii)  $\Omega(x') \in L(\Sigma)$  and its modulus of continuity  $\omega(\delta)$  satisfy the Dini condition:

$$\int_0^1 \frac{\omega(\delta)}{\delta} d\delta < \infty.$$

Then they proved that the operation  $Tf = \tilde{f}$  is linear and type  $(p, p)$  for every  $p > 1$ , weak type  $(1, 1)$  respectively.

Another one is that of Hardy-Littlewood-Sobolev and they considered the singular integral operator of potential type

$$\tilde{f}_\lambda(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\lambda}} dy, \quad (0 < \lambda < 1).$$

If we write  $1 < r < s < \infty$ ,  $1/r - 1/s = 1 - \lambda$ , then it is proved that this operator is of type  $(r, s)$  in the one-dimensional case by G.H. Hardy-J.E. Littlewood [2] in the  $n$ -dimensional case by S.L. Sobolev [8] respectively. A. Zygmund [10] also proved that it is of weak type  $(1, 1/\lambda)$  in the  $n$ -dimensional case.

From now on, if no confusion arises we shall omit the symbols of spaces and

measures  $\mu$  and  $\nu$  for norms. The author thanks to Prof. H. Tanabe for his sincere advices as a referee.

**2. Proof of Theorem 2.** We shall begin to prove auxiliary result.

**Proposition 1.** *Under the hypotheses of theorem 2, if  $h \in L_{\mu^1}^{a_1}(\log^+ L_{\mu})^{a_1/b_1}$  we have*

$$(1) \quad \left( \int_{|Th| \leq 1} |Th|^{b_2} d\nu \right)^{1/b_2} \leq KM^{b_1/b_2} (|h| [L_{\mu^1}^{a_1}])^{b_1/b_2}$$

and

$$(2) \quad \left( \int_{|Th| > 1} |Th|^{b_1} d\nu \right)^{1/b_1} \leq KM^{b_2/b_1} (|h| [L_{\mu^1}^{a_1}(\log^+ L_{\mu})^{a_1/b_1}])^{\gamma}$$

where  $K$  is a constant independent of  $M_1$ ,  $M_2$  and  $f$ ,  $M = \max(M_1, M_2, 1)$  and  $\gamma$  equals to  $\min(1, a_1 b_2 / a_2 b_1)$  or  $\max(1, a_1 b_2 / a_2 b_1)$  according to  $|h| \leq 1$  or  $|h| > 1$  respectively.

Proof of Proposition 1. Let us denote by  $n(y)$  the distribution function of  $Th$ , then we have

$$\begin{aligned} \int_{|Th| \leq 1} |Th|^{b_2} d\nu &= - \int_0^1 y^{b_2} dn(y) \\ &= -y^{b_2} n(y) \Big|_{y=0}^{y=1} + b_2 \int_0^1 y^{b_2-1} n(y) dy \\ &\leq b_2 \int_0^1 y^{b_2-1} \left( \frac{M_1}{y} |h| [L_{\mu^1}^{a_1}] \right)^{b_1} dy = \frac{b_2}{b_2 - b_1} M_1^{b_1} (|h| [L_{\mu^1}^{a_1}])^{b_1}. \end{aligned}$$

Next we have

$$\begin{aligned} \int_{|Th| > 1} |Th|^{b_1} d\nu &= - \int_1^{\infty} y^{b_1} dn(y) \\ &= -y^{b_1} n(y) \Big|_{y=1}^{y=\infty} + b_1 \int_1^{\infty} y^{b_1-1} n(y) dy. \end{aligned}$$

and

$$n(1) \leq M_1^{b_1} (|h| [L_{\mu^1}^{a_1}])^{b_1}.$$

Now we shall run on lines of A. Zygmund[10]. Let us decompose  $h$  into  $h_1 + h_2$  such that

$$h_2 = \begin{cases} h, & \text{if } |h| \leq z \\ e^{i \arg h} z, & \text{if } |h| > z \end{cases} \quad h_1 = h - h_2.$$

Here  $z$  is a positive number greater than 1 and will be determined later. Since  $h_2 \in L_{\mu^2}^{a_2}$  and  $h_1 \in L_{\mu^1}^{a_1}(\log^+ L_{\mu})^{a_1/b_1} \subset L_{\mu^1}^{a_1}$ , if we denote by  $n_i(y)$  ( $i=1, 2$ ) the distribution function of  $Th_i$  ( $i=1, 2$ ), then we have

$$\begin{aligned} n(y) &\leq n_1(y/2\kappa) + n_2(y/2\kappa) \\ &\leq M_1^{b_1}(y/2\kappa)^{-b_1}(|h_1| | [L_{\mu^1}^a] ]^{b_1} + M_2^{b_2}(y/2\kappa)^{-b_2}(|h_2| | [L_{\mu^2}^a] ]^{b_2} \end{aligned}$$

therefore

$$\begin{aligned} &b_1 \int_1^\infty y^{b_1-1} n(y) dy \\ &\leq b_1(2\kappa)^{b_1} M_1^{b_1} \int_1^\infty y^{-1} |h_1| | [L_{\mu^1}^a] ]^{b_1} dy + b_1(2\kappa)^{b_2} M_2^{b_2} \int_1^\infty y^{b_1-b_2-1} |h_2| | [L_{\mu^2}^a] ]^{b_2} dy \\ &= b_1(2\kappa)^{b_1} M_1^{b_1} I_1 + b_1(2\kappa)^{b_2} M_2^{b_2} I_2, \text{ say.} \end{aligned}$$

Let us also denote by  $m(y)$  and  $m_i(y)$  the distribution function of  $h$  and  $h_i$  ( $i=1, 2$ ) respectively. Since  $m_2(t)=m(t)$  if  $0 < t \leq z$ ,  $=m(z)$  if  $t > z$ , we have

$$\begin{aligned} I_2 &= \int_1^\infty y^{b_1-b_2-1} \left( - \int_0^\infty t^{a_2} dm_2(t) \right)^{k_2} dy, \quad k_2 = b_2/a_2 \\ &= \int_1^\infty y^{b_1-b_2-1} \left( - t^{a_2} m(t) \Big|_{t=0}^{t=z} + a_2 \int_0^z t^{a_2-1} m(t) dt \right)^{k_2} dy \\ &\leq \int_1^\infty y^{b_1-b_2-1} \left( a_2 \int_0^z t^{a_2-1} m(t) dt \right)^{k_2} dy \end{aligned}$$

and

$$I_2^{1/k_2} \leq \sup_x a_2 \int_1^\infty y^{b_1-b_2-1} \left( \int_0^z t^{a_2-1} m(t) dt \right) \chi(y) dy$$

for  $\int_1^\infty y^{b_1-b_2-1} \chi^{k_2'}(y) dy \leq 1$ , where  $1/k_2 + 1/k_2' = 1$ . If we put  $z = y^\xi$ ,  $\xi > 0$ , we get

$$\begin{aligned} &a_2 \int_1^\infty y^{b_1-b_2-1} \left( \int_0^z t^{a_2-1} m(t) dt \right) \chi(y) dy \\ &= a_2 \int_1^\infty t^{a_2-1} m(t) dt \int_{t^{1/\xi}}^\infty y^{b_1-b_2-1} \chi(y) dy \\ &\quad + a_2 \int_0^1 t^{a_2-1} m(t) dt \int_1^\infty y^{b_1-b_2-1} \chi(y) dy. \end{aligned}$$

If  $t \geq 1$ , we have

$$\begin{aligned} \int_{t^{1/\xi}}^\infty y^{b_1-b_2-1} \chi(y) dy &\leq \left( \int_{t^{1/\xi}}^\infty y^{b_1-b_2-1} dy \right)^{1/k_2} \left( \int_{t^{1/\xi}}^\infty y^{b_1-b_2-1} \chi^{k_2'}(y) dy \right)^{1/k_2'} \\ &\leq (b_2 - b_1)^{-1/k_2} t^{-(b_2-b_1)/\xi k_2}. \end{aligned}$$

If we write  $a_2 - 1 - (b_2 - b_1)/\xi k_2 = a_1 - 1$  and solve with respect to  $\xi$ ,

$$\xi = \frac{b_2 - b_1}{k_2(a_2 - a_1)} = \frac{a_2(b_2 - b_1)}{b_2(a_2 - a_1)} > 0$$

thus we get



$$\begin{aligned}
a_2 \int_1^\infty y^{b_1-b_2-1} \left( \int_0^z t^{a_2-1} m(t) dt \right) \chi(y) dy \\
\leq a_2 (b_2-b_1)^{-1/k_2} \int_0^\infty t^{a_1-1} m(t) dt = \frac{a_2}{a_1} (b_2-b_1)^{-1/k_2} (|h| [L_{\mu^1}^{a_1}])^{a_1}
\end{aligned}$$

and so

$$I_2 \leq \left( \frac{a_2}{a_1} \right)^{k_2} (b_2-b_1)^{-1} (|h| [L_{\mu^1}^{a_1}])^{a_1 k_2}$$

Next, since  $m_1(t) = m(t+z)$  for all  $t \geq 0$ , we have

$$\begin{aligned}
I_1 &= \int_1^\infty y^{-1} \left( - \int_0^\infty t^{a_1} dm_1(t) \right)^{k_1} dy, \quad k_1 = b_1/a_1 \\
&= \int_1^\infty y^{-1} \left( -t^{a_1} m_1(t) \Big|_{t=0}^{t=\infty} + a_1 \int_0^\infty t^{a_1-1} m_1(t) dt \right)^{k_1} dy \\
&= \int_1^\infty y^{-1} \left( a_1 \int_z^\infty (t-z)^{a_1-1} m(t) dt \right)^{k_1} dy \\
&\leq \int_1^\infty y^{-1} \left( a_1 \int_z^\infty t^{a_1-1} m(t) dt \right)^{k_1} dy
\end{aligned}$$

and

$$I_1^{1/k_1} \leq \sup_{\omega} a_1 \int_1^\infty y^{-1} \left( \int_z^\infty t^{a_1-1} m(t) dt \right) \omega(y) dy$$

for  $\int_1^\infty y^{-1} \omega^{k_1'}(y) dy \leq 1$ , where  $1/k_1 + 1/k_1' = 1$ . Since  $z = y^\xi$ ,  $\xi > 0$  we get

$$\begin{aligned}
a_1 \int_1^\infty y^{-1} \left( \int_z^\infty t^{a_1-1} m(t) dt \right) \omega(y) dy &= a_1 \int_1^\infty t^{a_1-1} m(t) dt \int_1^{t^{1/\xi}} y^{-1} \omega(y) dy \\
&\leq a_1 \int_1^\infty t^{a_1-1} m(t) dt \left( \int_1^{t^{1/\xi}} y^{-1} dy \right)^{1/k_1} \left( \int_1^{t^{1/\xi}} y^{-1} \omega^{k_1'}(y) dy \right)^{1/k_1'} \\
&= a_1 \int_1^\infty t^{a_1-1} (\log t^{1/\xi})^{1/k_1} m(t) dt = a_1 \xi^{-1/k_1} \int_1^\infty t^{a_1-1} (\log t)^{1/k_1} m(t) dt
\end{aligned}$$

and since

$$\begin{aligned}
a_1 t^{a_1-1} (\log t)^{1/k_1} &= (t^{a_1} (\log t)^{1/k_1})' - (1/k_1) t^{a_1-1} (\log t)^{1/k_1-1} \\
&< (t^{a_1} (\log t)^{1/k_1})', \quad \text{if } t > 1
\end{aligned}$$

The last formula does not exceed the following,

$$\begin{aligned}
&\xi^{-1/k_1} \int_1^\infty (t^{a_1} (\log t)^{1/k_1})' m(t) dt \\
&= \xi^{-1/k_1} \left( t^{a_1} (\log t)^{1/k_1} m(t) \Big|_{t=1}^{t=\infty} - \int_1^\infty t^{a_1} (\log t)^{1/k_1} dm(t) \right) \\
&= \xi^{-1/k_1} \int_{|h|>1} |h|^{a_1} (\log |h|)^{1/k_1} d\mu.
\end{aligned}$$

Therefore we get

$$I_1 \leq \xi^{-1} (||h|| [L_{\mu^1}^{\alpha_1} (\log^+ L_{\mu})^{\alpha_1/b_1}])^{b_1}$$

where  $\xi = a_2(b_2 - b_1)/b_2(a_2 - a_1) > 0$ . Summing up these estimations we get

$$\begin{aligned} \int_{|Th|>1} |Th|^{b_1} d\nu &\leq M_1^{b_1} (||h|| [L_{\mu^1}^{\alpha_1}])^{b_1} + b_1 (2\kappa)^{b_1} M_1^{b_1} \xi^{-1} \\ &\quad (||h|| [L_{\mu^1}^{\alpha_1} (\log^+ L_{\mu})^{\alpha_1/b_1}])^{b_1} \\ &\quad + b_1 (2\kappa)^{b_2} M_2^{b_2} \left(\frac{a_2}{a_1}\right)^{b_2} (b_2 - b_1)^{-1} (||h|| [L_{\mu^1}^{\alpha_1}])^{a_1 b_2}. \end{aligned}$$

We shall need the following lemma

**Lemma 1.** *Let us suppose the inequality  $A \leq \kappa(B+C)$  between three non-negative number  $A, B$  and  $C$ . Then we have the following inequalities:*

(i) if  $0 \leq A \leq 1$ ,

$$A \leq \begin{cases} \kappa(B+C), & \text{if } 0 \leq C \leq 1 \\ \kappa(B+C^p), & \text{if } C > 1 \end{cases}$$

(ii) if  $A > 1$ ,

$$A \leq \begin{cases} B^q + C^q, & \text{if } 0 \leq C \leq 1 \\ B^q + C, & \text{if } C > 1 \end{cases}$$

where  $p = b_1/b_2$  and  $q = b_2/b_1$  respectively.

End of proof of Theorem 2. We shall use a constant  $K$  depending only on  $\alpha_1, \alpha_2, \beta_1, \beta_2$  and  $\kappa$  and use the same letter at each occurrence. We shall also denote by  $M$  the maximum value of  $M_1, M_2$  and 1. If  $f \in L_{\mu^1}^{\alpha_1} (\log^+ L_{\mu})^{\alpha_1/b_1} + L_{\mu^2}^{\alpha_2}$ , for any positive number  $\eta$  there exists a decomposition of  $f = g + h$  with  $g \in L_{\mu^2}^{\alpha_2}, h \in L_{\mu^1}^{\alpha_1} (\log^+ L_{\mu})^{\alpha_1/b_1}$  such that  $||g|| + ||h|| \leq ||f|| + \eta$ . Here if  $||f|| < 1$  then we should take the  $\eta$  so small that  $||f|| + \eta \leq 1$ .

From now on we shall denote simply  $||f||, ||g||$  and  $||h||$  respectively. Let us denote by  $S_1$  and  $S_2$  the set of points  $|Tf| > 1$  and  $|Tf| \leq 1$  respectively. Let us also denote by  $S_{11}$  and  $S_{12}$  the set of points  $|Th| > 1$  and  $|Th| \leq 1$  respectively. Then applying the first part of lemma 1 with  $A = |Tf|, B = |Tg|, C = |Th|$  and integrating over  $S_2$ , we get

$$\begin{aligned} &\left( \int_{S_2} |Tf|^{b_2} d\nu \right)^{1/b_2} \\ &\leq \kappa \left( \int_{S_2 \cap S_{12}} (|Tg| + |Th|)^{b_2} d\nu \right)^{1/b_2} + \kappa \left( \int_{S_2 \cap S_{11}} (|Tg| + |Th|^p)^{b_2} d\nu \right)^{1/b_2} \\ &\leq 2\kappa \left( \int_S |Tg|^{b_2} d\nu \right)^{1/b_2} + \kappa \left( \int_{S_{12}} |Th|^{b_2} d\nu \right)^{1/b_2} + \kappa \left( \int_{S_{11}} |Th|^{b_1} d\nu \right)^{1/b_2} \end{aligned}$$

by the Minkowsky inequality. Since  $g \in L_{\mu^2}^{a_2}$  we have by hypotheses

$$(3) \quad ||Tg||[L_{\nu^2}^{b_2}] \leq M_2 ||g||[L_{\mu^2}^{a_2}].$$

If we substitute estimations (1), (2) and (3), we get

$$\left( \int_{S_2} |Tg|^{b_2} d\nu \right)^{1/b_2} \leq KM(||f|| + \eta)^{\gamma_1}$$

where the index  $\gamma_1$  equals to  $\min(b_1/b_3, a_1/a_2)$  or 1 according to  $||f|| < 1$  or  $||f|| \geq 1$  respectively. Next if we apply the second part of lemma 1, we get by repetitions of the same discussion

$$\begin{aligned} & \left( \int_{S_1} |Tf|^{b_1} d\nu \right)^{1/b_1} \\ & \leq \left( \int_{S_1 \cap S_{12}} (|Tg|^q + |Th|^q)^{b_1} d\nu \right)^{1/b_1} + \left( \int_{S_1 \cap S_{11}} (|Tg|^q + |Th|^{b_1} d\nu \right)^{1/b_1} \\ & \leq 2 \left( \int_S |Tg|^{b_2} d\nu \right)^{1/b_1} + \left( \int_{S_{12}} |Th|^{b_2} d\nu \right)^{1/b_1} + \left( \int_{S_{11}} |Th|^{b_1} d\nu \right)^{1/b_1} \\ & \leq KM^{b_2/b_1} (||f|| + \eta)^{\gamma_2} \end{aligned}$$

where the index  $\gamma_3$  equals to  $\min(1, a_1 b_2 / a_2 b_1)$  or  $b_2/b_1$  according to  $||f|| < 1$  or  $||f|| \geq 1$  respectively.

Let us first suppose that  $||f|| = 1$ , then by the above estimations we have if  $\eta$  tends to zero,

$$\left( \int_{|Tf| \leq 1} |Tf|^{b_2} d\nu \right)^{1/b_2} + \left( \int_{|Tf| > 1} |Tf|^{b_1} d\nu \right)^{1/b_1} \leq KM^{b_2/b_1}$$

thus we have proved

$$(4) \quad ||Tf|| \leq KM^{b_2/b_1} ||f||, \quad M = \max(M_1, M_2, 1).$$

Now we shall exclude the assumption  $||f|| = 1$ . We use the properties as for  $||f||$  which is neither norm nor quasi-norm and so we shall prove these for the sake of completeness.

**Lemma 2.** *The pseudo-norm  $||f||[L_{\mu^1}^{a_1}(\log^+ L_{\mu})^{a_1/b_1} + L_{\mu^2}^{a_2}]$  has the following properties:*

(i) *if  $\lambda$  is any positive real number, it is satisfied*

$$2^{-1/b_1} \lambda \{1 + (\log^+ \lambda^{-1})^{a_1/b_1}\}^{-1/a_1} ||f|| \leq ||\lambda f|| \leq 2^{1/b_1} \lambda \{1 + (\log^+ \lambda)^{a_1/b_1}\}^{1/a_1} ||f||$$

(ii) *The pseudo-norm  $||\lambda f||$  is a continuous function of  $\lambda$ .*

Proof of Lemma 2. (i) Since for any positive number  $\lambda$

$$\begin{aligned} 1 + (\log^+ |\lambda h|)^{a_1/b_1} &\leq 1 + (\log^+ \lambda + \log^+ |h|)^{a_1/b_1} \\ &\leq 2^{a_1/b_1} \{1 + (\log^+ \lambda)^{a_1/b_1}\} \{1 + (\log^+ |h|)^{a_1/b_1}\} \end{aligned}$$

we have by the decomposition of  $f = g + h$ ,

$$\begin{aligned} ||\lambda f|| &\leq ||\lambda g|| + ||\lambda h|| \leq \lambda ||g|| + 2^{1/b_1} \lambda \{1 + (\log^+ \lambda)^{a_1/b_1}\}^{1/a_1} ||h|| \\ &\leq 2^{1/b_1} \lambda \{1 + (\log^+ \lambda)^{a_1/b_1}\}^{1/a_1} (||f|| + \eta). \end{aligned}$$

If we let  $\eta$  tend to 0, the second inequality of (i) has proved. Next in this inequality, let us put  $\lambda^{-1}f$  instead of  $f$ .

$$||f|| \leq 2^{1/b_1} \lambda \{1 + (\log^+ \lambda)^{a_1/b_1}\}^{1/a_1} ||\lambda^{-1}f||$$

and write  $\lambda^{-1} = \mu$ , we have

$$2^{-a_1/b_1} \mu \{1 + (\log^+ \mu^{-1})^{a_1/b_1}\}^{-1/a_1} ||f|| \leq ||\mu f||$$

This is just the first inequality of (i).

(ii) Let us suppose that  $\lambda_n \rightarrow \lambda$ . The case  $\lambda = 0$  is trivial from the inequality (i). We suppose that  $\lambda > 0$ . For any given positive number, there exists a decomposition  $\lambda f = G + H$ ,  $G \in L_{\mu^2}^{a_2}$ ,  $H \in L_{\mu^2}^{a_2} (\log^+ L_{\mu})^{a_1/b_1}$  such that  $||G|| + ||H|| \leq ||\lambda f|| + \eta$ . If we put  $G = \lambda g$ ,  $H = \lambda h$ , then  $g + h = f$  and so  $||\lambda_n f|| \leq ||\lambda_n g|| + ||\lambda_n h||$ . Both  $||\lambda g||$  and  $||\lambda h||$  are continuous with respect to  $\lambda$ , therefore

$$\begin{aligned} \lim_{n \rightarrow \infty} ||\lambda_n f|| &\leq \lim_{n \rightarrow \infty} \{||\lambda_n g|| + ||\lambda_n h||\} \\ &= ||\lambda g|| + ||\lambda h|| = ||G|| + ||H|| \leq ||\lambda f|| + \eta. \end{aligned}$$

Next we shall prove that  $||\lambda f|| \leq \lim_{n \rightarrow \infty} ||\lambda_n f||$ . Since  $||\lambda f||$  is a monotone non-decreasing function with respect to  $\lambda > 0$ , we can assume that  $\lambda_n < \lambda$ . We have a similar decomposition  $\lambda_n f = G_n + H_n$  such that  $G_n \in L_{\mu^2}^{a_2}$ ,  $H_n \in L_{\mu^2}^{a_2} (\log^+ L_{\mu})^{a_1/b_1}$  and  $||G_n|| + ||H_n|| \leq ||\lambda_n f|| + \eta$ . If we write  $G_n = \lambda_n g_n$ ,  $H_n = \lambda_n h_n$ , then  $g_n + h_n = f$  for all  $n$ .

Therefore we get

$$\begin{aligned} ||\lambda f|| &\leq ||\lambda g_n|| + ||\lambda h_n|| \\ &\leq ||G_n|| + ||H_n|| + (||\lambda g_n|| - ||\lambda_n g_n||) + (||\lambda h_n|| - ||\lambda_n h_n||) \\ &\leq ||\lambda_n f|| + (||\lambda g_n|| - ||\lambda_n g_n||) + (||\lambda h_n|| - ||\lambda_n h_n||) + \eta \end{aligned}$$

where we can prove easily

$$\begin{aligned} ||\lambda g_n|| - ||\lambda_n g_n|| &\rightarrow 0 \quad (n \rightarrow \infty) \\ ||\lambda h_n|| - ||\lambda_n h_n|| &\rightarrow 0 \quad (n \rightarrow \infty) \end{aligned}$$

by the use of  $||G_n|| \leq ||\lambda f|| + \eta$  and  $||H_n|| \leq ||\lambda f|| + \eta$  for sufficiently large  $n$ . Thus we obtain  $||\lambda f|| \leq \lim_{n \rightarrow \infty} ||\lambda_n f||$ .

From lemma 2, we can conclude that if  $||f|| \neq 1$ , there exist a positive real number  $\lambda$  such that  $||\lambda f|| = 1$ . If  $||f|| > 1$ , then the  $\lambda$  to make  $||f|| = 1$  is less than 1 and on the contrary if  $||f|| < 1$  then the  $\lambda$  is greater than 1. Since  $\tilde{f} = Tf$  is a quasi-linear operation and  $L_{\nu^1}^{b_1} + L_{\nu^2}^{b_2}$  is a Banach space, we have

$$(5) \quad ||T(\lambda f)|| [L_{\nu^1}^{b_1} + L_{\nu^2}^{b_2}] = \lambda ||Tf|| [L_{\nu^1}^{b_1} + L_{\nu^2}^{b_2}].$$

If  $||f|| > 1$ , then we use the inequality (4) with respect to  $\lambda f$  such that  $||\lambda f|| = 1$  and applying the second part of lemma 2 (i) and (5), we can derive that the inequality (4) is also true. If  $||f|| < 1$  then we use the inequality (4) with respect to  $\lambda f$  such that  $||\lambda f|| = 1$  again and applying the second part of lemma 2, (i) and (5) we have

$$||Tf|| \leq 2^{1/b_1} M^{b_2/b_1} \{1 + (\log^+ \lambda)^{a_1/b_1}\}^{1/a_1} ||f||.$$

Since  $2^{-1/b_1} \lambda ||f|| \leq ||\lambda f|| = 1$  by the first part of lemma 2, (i) we can derive that

$$(6) \quad ||Tf|| \leq K M^{b_2/b_1} ||f|| \{1 + (\log^+ ||f||^{-1})^{a_1/b_1}\}^{1/a_1}.$$

Thus theorem 2 has proved completely.

**3. Proofs of Theorems 3 and 4.** Since the method of proofs of theorems 3 and 4 is just the same as the preceding section, we only sketch the outlines.

**Proposition 2.** *Let us suppose that the quasi-linear operation  $T$  is of weak type  $(a_1, b_1)$ ,  $1 \leq a_1$ ,  $b_1 < \infty$ . Then if  $h \in L_{\mu^1}^{a_1}$  we have*

$$(7) \quad \left( \int_{|Th| > 1} \frac{|Th|^{b_1}}{1 + (\log |Th|)^{1+\varepsilon}} d\nu \right)^{1/b_1} \leq K M_1 ||h|| [L_{\nu^1}^{a_1}],$$

and

$$(8) \quad \left( \int_{|Th| \leq 1} \frac{|Th|^{b_1}}{1 + (\log |Th|^{-1})^{1+\varepsilon}} d\nu \right)^{1/b_1} \leq K M_1 ||h|| [L_{\mu^1}^{a_1}],$$

where  $\varepsilon$  is any positive number.

**Proof of Proposition 2.** We have by the same way as the proof of proposition 1,

$$\begin{aligned} \int_{|Th| > 1} \frac{|Th|^{b_1}}{1 + (\log |Th|)^{1+\varepsilon}} d\nu &= - \int^\infty \frac{y^{b_1}}{1 + (\log y)^{1+\varepsilon}} dn(y) \\ &= - \frac{y^{b_1}}{1 + (\log y)^{1+\varepsilon}} n(y) \Big|_{y=1}^{y=\infty} + \int_1^\infty \left( \frac{y^{b_1}}{1 + (\log y)^{1+\varepsilon}} \right)' n(y) dy \end{aligned}$$

If  $y > 1$ , we have

$$\begin{aligned} \left( \frac{y^{b_1}}{1 + (\log y)^{1+\varepsilon}} \right)' &= \frac{b_1 y^{b_1-1} \{1 + (\log y)^{1+\varepsilon}\} - (1+\varepsilon) y^{b_1-1} (\log y)^\varepsilon}{\{1 + (\log y)^{1+\varepsilon}\}^2} \\ &\leq b_1 y^{b_1-1} / \{1 + (\log y)^{1+\varepsilon}\} \end{aligned}$$

and therefore

$$\begin{aligned} \int_{|Th|>1} \frac{|Th|^{b_1}}{1 + (\log |Th|)^{1+\varepsilon}} d\nu &\leq (M_1 |h|)^{b_1} + b_1 \int_1^\infty \frac{y^{b_1-1}}{1 + (\log y)^{1+\varepsilon}} n(y) dy \\ &\leq (M_1 |h|)^{b_1} + b_1 M_1^{b_1} \int_1^\infty \frac{dy}{y \{1 + (\log y)^{1+\varepsilon}\}} |h|^{b_1} \leq K M_1^{b_1} |h|^{b_1} \end{aligned}$$

The remaining part is proved by the same way.

For the proof of theorem 3, we need the following lemmas.

**Lemma 3.** *From an inequality  $A \leq \kappa(B+C)$  between three non-negative numbers  $A$ ,  $B$  and  $C$ , we can derive the following*

(i) if  $0 \leq A \leq 1$ ,

$$A \leq \begin{cases} \kappa(B+C), & \text{if } 0 \leq C \leq 1 \\ \text{const.} \left( B + \left( \frac{C^{b_1}}{1 + (\log C)^{1+\varepsilon}} \right)^{1/b_2} \right), & \text{if } C > 1 \end{cases}$$

(ii) if  $A > 1$ ,

$$\frac{A^{b_1}}{1 + (\log A)^{1+\varepsilon}} \leq \begin{cases} (2\kappa)^{b_2} (B^{b_2} + C^{b_2}), & \text{if } 0 \leq C \leq 1. \\ \text{const.} \left( B^{b_2} + \frac{C^{b_1}}{1 + (\log C)^{1+\varepsilon}} \right), & \text{if } C > 1 \end{cases}$$

**Lemma 4.** *The pseudo-norm  $\|f\| [L_{\nu}^{b_1}] (\log^+ L_{\nu})^{1+\varepsilon} + L_{\nu}^{b_2}$  satisfy the following inequality: there exists a constant  $C$  such that*

$$\frac{\lambda}{C \{1 + (\log^+ \lambda)^{(1+\varepsilon)/b_1}\}} \|f\| \leq \|\lambda f\| \leq C \lambda \{1 + (\log^+ \lambda)^{(1+\varepsilon)/b_1}\} \|f\|.$$

If we repeat the discussion of the proof of theorem 2, we can prove theorem 3. The theorem 4 is an immediate consequence of proposition 2.

**4. Proofs of Theorem 5, 6 and 7.** We shall need the following proposition and lemma.

**Proposition 3.** *Let us suppose that the quasi-linear operation  $T$  is of weak type  $(a_1, b_1)$ ,  $1 \leq a_1, b_1 < \infty$ . Then if  $h \in L_{\mu}^{a_1}$  we have*

$$(9) \quad \left( \int_{|Th|>1} |Th|^{b_1-\varepsilon} d\nu \right)^{1/b_1-\varepsilon} \leq K M_1^{b_1/b_1-\varepsilon} (\|h\| [L_{\mu}^{a_1}]^{b_1/b_1-\varepsilon})$$

and

$$(10) \quad \left( \int_{|Th| \leq 1} |Th|^{b_1+\varepsilon} d\nu \right)^{1/b_1+\varepsilon} \leq KM_1^{b_1/b_1+\varepsilon} (|h| [L_\mu^{\alpha_1}])^{b_1/b_1+\varepsilon}$$

where  $\varepsilon$  is any positive number.

**Lemma 5.** From an inequality  $A \leq \kappa(B+C)$  between three non-negative numbers  $A$ ,  $B$  and  $C$ , we can derive the following

(i) if  $0 \leq A \leq 1$ ,

$$A \leq \begin{cases} \kappa(B+C), & \text{if } 0 \leq C \leq 1 \\ B+C^{b_1^{-\varepsilon/b_2}}, & \text{if } C > 1 \end{cases}$$

(ii) if  $A > 1$ ,

$$A \leq \begin{cases} (2\kappa)^{b_2/b_1-\varepsilon} (B^{b_2/b_1-\varepsilon} + C^{b_2/b_1-\varepsilon}), & \text{if } 0 \leq C \leq 1 \\ (2\kappa)^{b_2/b_1-\varepsilon} (B^{b_2/b_1-\varepsilon} + C), & \text{if } C > 1. \end{cases}$$

Now proofs of these theorems are repetitions of those of preceding section and need not be gone into the details.

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